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Lattice animals on a staircase and Fibonacci numbers

L Turban

Laboratoire de Physique des Matériaux, Université Henri Poincaré (Nancy I), BP 239,
F–54506 Vandœuvre lès Nancy Cedex, France

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Abstract. We study the statistics of column-convex lattice animals resulting from the stacking of squares on a single or double staircase. We obtain exact expressions for the number of animals with a given length and area, their mean length and their mean height. These objects are closely related to Fibonacci numbers. On a single staircase, the total number of animals with area k is given by the Fibonacci number F_k.

1. Introduction

A *lattice animal* is a cluster of occupied sites on a lattice. In two dimensions, the *area* of an animal is the number of sites belonging the cluster and its *perimeter* is defined as the set of empty first neighbours of occupied sites. Alternatively, instead of the site clusters, one may consider the corresponding clusters of occupied cells on the dual lattice, also called *polyominoes*.

The enumeration of lattice animals according to their area and/or perimeter is a subject of active research in the statistical physics and combinatorics communities (see [1] for a recent review). On the physical side, the main interest lies in the close connection between lattice animals and the percolation problem [2, 3].

In the most general case, the counting problem is quite difficult and only some bounds on the asymptotic behaviour are known [4]. This led to the introduction of restricted classes of animals (Ferrers graphs, convex and/or directed animals) for which some exact results could be obtained, mainly in two dimensions (see for example [5–12]).

In this paper, we consider animals resulting from the stacking of squares on a single or double staircase, i.e. column-convex (or vertically convex) animals, for which the intersection of a vertical line with the perimeter has at most two connected components. These animals are closely related to Fibonacci numbers. In the case of a single staircase, the correspondance is particularly simple: the area-generating function is *equal* to the Fibonacci number generating function.

The paper is organized as follows. In section 2, we study the stacking of squares on a single staircase. We calculate the number of animals with a given length and area, the total number of animals with a given area, their mean length and mean height and study the asymptotic behaviours. The same is done in section 3 for animals on a double staircase. The results are discussed in section 4. Some technical details are given in the appendix.

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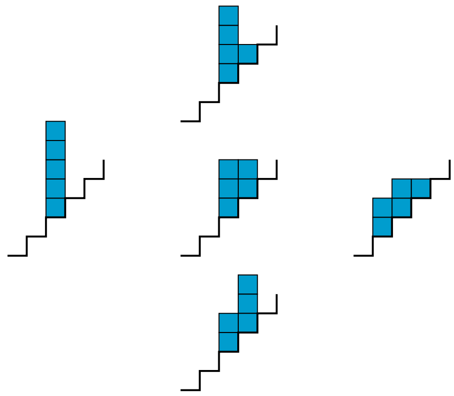

Figure 1. Different stackings of five squares on a staircase arranged by number of occupied stairs.

2. Single staircase

In this section we consider column-convex lattice animals resulting from the stacking of k squares on a single staircase as shown in figure 1. Two neighbouring columns are connected when they share at least one edge.

2.1. Number of animals

In order to count the number $F_{k,l}$ of different animals with area k living on l stairs, we introduce the generating function

$$F(z,t) = \sum_{k,l=1}^{\infty} F_{k,l} z^k t^l. \tag{2.1}$$

It satisfies the relation

$$F(z,t) = zt + t\frac{z^2}{1-z}[1 + F(z,t)] \tag{2.2}$$

where the first term corresponds to a single square, the factor $z^2/(1-z)$ in the second term is the generating function of a column with at least two squares, needed for the animal to eventually continue its growth on the next stair. Hence we have

$$\begin{aligned} F(z,t) &= \frac{zt}{1-z-tz^2} = zt\sum_{n=0}^{\infty} z^n(1+zt)^n \\ &= \sum_{n=0}^{\infty}\sum_{p}\binom{n}{p} z^{n+p+1}t^{p+1} = \sum_{l}\sum_{k=l}^{\infty}\binom{k-l}{l-1} z^k t^l. \end{aligned} \tag{2.3}$$

By convention, in sums containing binomial coefficients, the range of summation is not explicitly indicated. It is automatically determined by the nonvanishing values of the binomial coefficients and, for example, $0 \leqslant p \leqslant n$ in equation (2.3).

The identification of the coefficients of $z^k t^l$ in equations (2.1) and (2.3) leads to

$$F_{k,l} = \binom{k-l}{l-1}. \tag{2.4}$$

Using the addition/induction relation for the binomial coefficients [13, p 174],

$$\binom{k-l+1}{l-1} = \binom{k-l}{l-1} + \binom{k-l}{l-2} \tag{2.5}$$

one obtains the recursion relation:

$$F_{k+1,l} = F_{k,l} + F_{k-1,l-1}. \tag{2.6}$$

The total number of animals with area k is given by

$$F_k = \sum_l \binom{k-l}{l-1} \tag{2.7}$$

i.e. by the kth Fibonacci number (see [13, p 303]) which satisfies the recursion relation $F_{k+1} = F_k + F_{k-1}$ with the initial values $F_0 = 0$ and $F_1 = 1$, according to (2.6) and (2.7).

This relation with the Fibonacci numbers† can be deduced directly by setting $t = 1$ in the first line of (2.3) which leads to

$$F(z,1) = F(z) = \sum_{k=1}^{\infty} F_k z^k = \frac{z}{1-z-z^2} \tag{2.8}$$

where $F(z)$ is the generating function of the Fibonacci numbers.

All the animals built from five squares are shown in figure 1, the different columns corresponding to the different values of l.

2.2. *Mean length*

The mean length of animals with area k is defined as

$$\bar{l}(k) = \frac{\sum_{l=1}^{\infty} l F_{k,l}}{\sum_{l=1}^{\infty} F_{k,l}} = \frac{A_k}{F_k} \qquad A_k = \sum_l l \binom{k-l}{l-1}. \tag{2.9}$$

In order to calculate A_k, let us introduce the auxiliary generating function

$$A(z) = \sum_{k=1}^{\infty} A_k z^k = \left.\frac{\partial F(z,t)}{\partial t}\right|_{t=1} = F(z) + zF^2(z) = \frac{1-z}{z} F^2(z). \tag{2.10}$$

According to [13, p 354]

$$F^2(z) = \sum_{k=2}^{\infty} F_k^{(2)} z^k \qquad F_k^{(2)} = \sum_{m+n=k} F_n F_m = \frac{2kF_{k+1} - (k+1)F_k}{5} \tag{2.11}$$

hence the coefficients of $A(z)$ are

$$A_k = [z^k](z^{-1}-1)F^2(z) = F_{k+1}^{(2)} - F_k^{(2)} = \frac{k(2F_k - F_{k-1}) + 3F_k}{5} \tag{2.12}$$

where we used the recursion relation for the Fibonacci numbers. The symbol $[z^k]f(z)$ means the coefficient of z^k in the series $f(z)$. The mean length follows from equations (2.9) and (2.12) and reads

$$\bar{l}(k) = \frac{k}{5}\left(2 - \frac{F_{k-1}}{F_k}\right) + \frac{3}{5}. \tag{2.13}$$

† A similar connection between the number of column-convex *directed* animals with area k and the Fibonacci numbers with *odd* indices F_{2k-1} has been noticed a long time ago. See [14].

The same method allows a calculation of the mean-square deviation of the length of animals with area k. The generating function

$$C(z) = \sum_{k=1}^{\infty} C_k z^k \qquad C_k = \sum_l l(l-1)\binom{k-l}{l-1} \tag{2.14}$$

is given by the second derivative of $F(z,t)$ at $t=1$,

$$C(z) = \left.\frac{\partial^2 F(z,t)}{\partial t^2}\right|_{t=1} = 2zF^2(z) + 2z^2F^3(z). \tag{2.15}$$

The coefficients of the series $F^3(z)$ are needed to obtain C_k. They are calculated in the appendix with the following result:

$$F^3(z) = \sum_{k=3}^{\infty} F_k^{(3)} z^k \qquad F_k^{(3)} = \frac{(k+1)(k+2)}{10} F_k - \frac{3}{5} F_{k+1}^{(2)}. \tag{2.16}$$

Thus we have

$$C_k = 2F_{k-1}^{(2)} + 2F_{k-2}^{(3)} = 4\left[\frac{2(k-1)F_k - kF_{k-1}}{25}\right] + \frac{k(k-1)}{5} F_{k-2} \tag{2.17}$$

and, using (2.12),

$$\overline{l^2}(k) = \frac{C_k + A_k}{F_k} = \frac{1}{25}\left[5k^2\left(1 - \frac{F_{k-1}}{F_k}\right) + k\left(13 - 4\frac{F_{k-1}}{F_k}\right) + 7\right]. \tag{2.18}$$

The mean-square deviation $\overline{\Delta l^2}(k) = \overline{l^2}(k) - \overline{l}^2(k)$ follows from (2.13) and (2.18).

2.3. Mean height

The mean height of an animal with area k living on l stairs is taken as the ratio k/l, i.e. it is measured from the staircase. Thus the mean height of animals with area k is given by

$$\overline{h}(k) = \frac{\sum_{l=1}^{\infty} kl^{-1} F_{k,l}}{\sum_l F_{k,l}} = \frac{B_k}{F_k} \qquad B_k = \sum_l \frac{k}{l}\binom{k-l}{l-1}. \tag{2.19}$$

The absorption/extraction identity can be used to write

$$\binom{k-l+1}{l} = \frac{k-l+1}{l}\binom{k-l}{l-1} = \frac{k+1}{l}\binom{k-l}{l-1} - \binom{k-l}{l-1} \qquad l \geqslant 1 \tag{2.20}$$

from which we deduce:

$$\frac{k}{l}\binom{k-l}{l-1} = \frac{k}{k+1}\left[\binom{k-l+1}{l} + \binom{k-l}{l-1}\right] \qquad l \geqslant 1. \tag{2.21}$$

Inserting (2.21) into the definition of B_k in (2.19), we obtain

$$\begin{aligned} B_k &= \frac{k}{k+1}\left[\sum_{l\geqslant 1}\binom{k-l+1}{l} + \sum_l \binom{k-l}{l-1}\right] \\ &= \frac{k}{k+1}\left[\sum_l \binom{k-l+1}{l} - 1 + \sum_l \binom{k-l}{l-1}\right] \end{aligned} \tag{2.22}$$

where the last equation follows from adding and substracting the term $l = 0$ in the first sum. According to the combinatorial definition of the Fibonacci numbers in (2.7) we have

$$B_k = \frac{k}{k+1}(F_{k+2} + F_k - 1) \tag{2.23}$$

and finally

$$\overline{h}(k) = \frac{k}{k+1}\left(\frac{F_{k+2}}{F_k} + 1 - \frac{1}{F_k}\right). \tag{2.24}$$

2.4. Asymptotic behaviour

The generating function of the Fibonacci numbers in (2.8) can be written as the partial fraction expansion

$$F(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right) \qquad \phi = \frac{1+\sqrt{5}}{2} \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2} \tag{2.25}$$

where $\phi \approx 1.618\,03$ is the *golden ratio* and $\hat{\phi} = 1 - \phi$. Thus the Fibonacci numbers are given by

$$F_k = \frac{1}{\sqrt{5}}(\phi^k - \hat{\phi}^k). \tag{2.26}$$

Generally the number of animals with size k behaves at large size as $Ck^{-\theta}\lambda^k$, where C is a constant amplitude, λ is the inverse of the critical fugacity z_c and θ is the exponent governing the critical behaviour of the generating function $G(z, 1) \sim (z_c - z)^{\theta - 1}$ when $z \to z_{c-}$. From (2.7) and (2.26) we deduce the critical parameters:

$$\theta = 0 \qquad \lambda = z_c^{-1} = \phi. \tag{2.27}$$

According to equations (2.13) and (2.26), the mean length behaves asymptotically as

$$\bar{l}(k) = \frac{2+\hat{\phi}}{5}k + \mathrm{O}(1) = \frac{5-\sqrt{5}}{10}k + \mathrm{O}(1). \tag{2.28}$$

For the mean-square length we have

$$\overline{l^2}(k) = \frac{1+\hat{\phi}}{5}k^2 + \frac{13+4\hat{\phi}}{25}k + \mathrm{O}(1) \tag{2.29}$$

whereas

$$\bar{l}(k)^2 = \frac{(2+\hat{\phi})^2}{25}k^2 + \frac{6(2+\hat{\phi})}{25}k + \mathrm{O}(1). \tag{2.30}$$

It is easy to verify that the leading $\mathrm{O}(k^2)$ contributions in (2.29) and (2.30) are the same. Thus, the mean-square deviation is $\mathrm{O}(k)$ and given by

$$\overline{\Delta l^2}(k) = \frac{k}{5\sqrt{5}} + \mathrm{O}(1). \tag{2.31}$$

The behaviour of the mean height follows from equations (2.24) and (2.26). It tends to a constant value:

$$\bar{h}(k) = 1 + \phi^2 + \mathrm{O}(k^{-1}) = \frac{5+\sqrt{5}}{2} + \mathrm{O}(k^{-1}). \tag{2.32}$$

3. Double staircase

Next we consider column-convex animals on a double staircase as shown in figure 2. The connectivity rules for neighbouring columns are the same as above for a single staircase and we count the different stackings with at least one square on the lowest central stair.

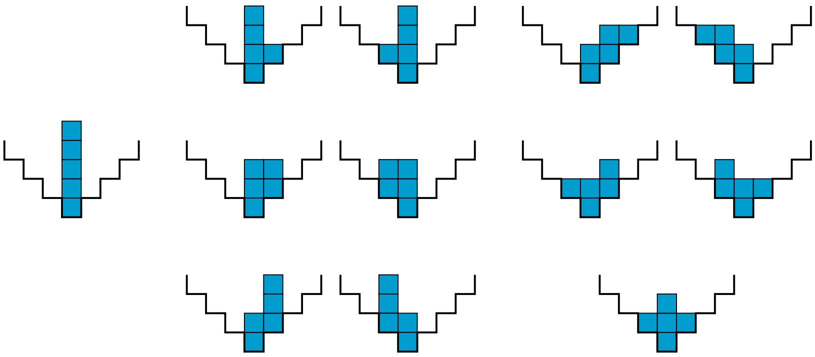

Figure 2. Different stackings of five squares on a double staircase arranged by number of occupied stairs.

3.1. Number of animals

The generating function for the number $G_{k,l}$ of animals with area k living on l successive stairs,

$$G(z,t) = \sum_{k,l=1}^{\infty} G_{k,l} z^k t^l \tag{3.1}$$

can be expressed using the generating function for animals in a single staircase in equation (2.3) as

$$G(z,t) = zt + t\frac{z^2}{1-z}[1 + F(z,t)]^2. \tag{3.2}$$

The first term corresponds to a single square on the central stair. When there are two or more squares on the central stair (first factor in the second term) the animal may either stop or continue to grow on one or two staircases (second factor in the second term). Using (2.2), the generating function may be rewritten as

$$G(z,t) = zt + [1 + F(z,t)][F(z,t) - zt] = (1 - zt)F(z,t) + F^2(z,t). \tag{3.3}$$

Accordingly, we have

$$\begin{aligned} G_{k,l} &= [z^k t^l]F(z,t) - [z^{k-1}t^{l-1}]F(z,t) + [z^k t^l]F^2(z,t) \\ &= \binom{k-l}{l-1} - \binom{k-l}{l-2} + \sum_{p,q} \binom{p-q}{q-1}\binom{k-p-l+q}{l-q-1}. \end{aligned} \tag{3.4}$$

Using the relation [13, p 169]

$$\sum_{p} \binom{p-q}{q-1}\binom{k-l-p+q}{l-q-1} = \binom{k-l+1}{l-1} \tag{3.5}$$

the double sum in (3.4) can be reduced to

$$\sum_{q=1}^{l-1}\sum_{p} \binom{p-q}{q-1}\binom{k-p-l+q}{l-q-1} = \sum_{q=1}^{l-1}\binom{k-l+1}{l-1} = (l-1)\binom{k-l+1}{l-1} \tag{3.6}$$

yielding

$$G_{k,l} = \binom{k-l}{l-1} - \binom{k-l}{l-2} + (l-1)\binom{k-l+1}{l-1} = l\binom{k-l+1}{l-1} - 2\binom{k-l}{l-2} \tag{3.7}$$

where the last expression follows from the addition/induction relation (2.5).

The total number of animals with area k reads as

$$G_k = \sum_l G_{k,l} = \sum_l l\binom{k-l+1}{l-1} - 2\sum_l \binom{k-l}{l-2}. \tag{3.8}$$

According to (2.9), the first term is equal to A_{k+1}, whereas the second follows from (2.7). Using the expression of A_k in (2.12) and the recursion on the Fibonacci numbers, we obtain

$$G_k = A_{k+1} - 2F_{k-1} = \frac{k(F_k + 2F_{k-1}) + 4F_k - 5F_{k-1}}{5}. \tag{3.9}$$

3.2. Mean length

As above, the mean length of animals with area k is defined as

$$\bar{l}(k) = \frac{\sum_{l=1}^{\infty} lG_{k,l}}{\sum_{l=1}^{\infty} G_{k,l}} = \frac{\mathcal{A}_k}{G_k} \qquad \mathcal{A}_k = \sum_l l^2\binom{k-l+1}{l-1} - 2\sum_l l\binom{k-l}{l-2}. \tag{3.10}$$

The $\mathcal{A}_k$s are the coefficients of the generating function

$$\begin{aligned}\mathcal{A}(z) &= \sum_{k=1}^{\infty} \mathcal{A}_k z^k = \left.\frac{\partial G(z,t)}{\partial t}\right|_{t=1} \\ &= -zF(z) + (1-z)A(z) + 2F(z)A(z) \\ &= -zF(z) + \frac{(1-z)^2}{z}F^2(z) + 2\frac{1-z}{z}F^3(z)\end{aligned} \tag{3.11}$$

which follows from (3.3), taking into account the expression of $A(z)$ given in (2.10).

Thus, according to (3.11), (2.11) and (2.16), we have

$$\begin{aligned}\mathcal{A}_k &= [z^k]\mathcal{A}(z) = -F_{k-1} + F_{k-1}^{(2)} - 2F_k^{(2)} + F_{k+1}^{(2)} + 2(F_{k+1}^{(3)} - F_k^{(3)}) \\ &= \frac{5k^2F_{k-1} + k(19F_k - 7F_{k-1}) + 6F_k - 25F_{k-1}}{25}.\end{aligned} \tag{3.12}$$

The mean length follows from equations (3.9), (3.10) and (3.12):

$$\bar{l}(k) = \frac{1}{5}\frac{5k^2F_{k-1} + k(19F_k - 7F_{k-1}) + 6F_k - 25F_{k-1}}{k(F_k + 2F_{k-1}) + 4F_k - 5F_{k-1}}. \tag{3.13}$$

The mean-square deviation can be determined, proceeding as above for the single staircase. It involves a lengthy calculation of the generating function $F^4(z)$, which can be obtained in the same way as $F^3(z)$ in the appendix. The asymptotic behaviour is the same as for the single staircase as shown in the discussion.

3.3. Mean height

The mean height, measured from the staircase, takes the form

$$\bar{h}(k) = \frac{\sum_{k=1}^{\infty} kl^{-1}G_{k,l}}{\sum_{k=1}^{\infty} G_{k,l}} = \frac{\mathcal{B}_k}{G_k} \qquad \mathcal{B}_k = k\sum_l \binom{k-l+1}{l-1} - 2\sum_l \frac{k}{l}\binom{k-l}{l-2}. \tag{3.14}$$

According to (2.7), the first sum is equal to F_{k+1}. Using (2.5) and (2.19), the second sum can be rewritten as

$$\sum_l \frac{k}{l}\binom{k-l}{l-2} = \sum_l \frac{k}{l}\binom{k-l+1}{l-1} - \sum_l \frac{k}{l}\binom{k-l}{l-1} = \frac{k}{k+1}B_{k+1} - B_k. \tag{3.15}$$

Collecting these results and taking (2.23) into account, we obtain

$$\mathcal{B}_k = kF_{k+1} + \frac{2k}{k+1}(F_{k+1} + 2F_k - 1) - \frac{2k}{k+2}(3F_{k+1} + F_k - 1)$$
$$= \frac{k}{(k+1)(k+2)}[k(k-1)F_{k+1} + 2(k+3)F_k - 2]. \tag{3.16}$$

Finally, the mean height follows from (3.9) and (3.16) and reads

$$\overline{h}(k) = \frac{5k}{(k+1)(k+2)} \frac{k^2(F_k + F_{k-1}) + k(F_k - F_{k-1}) + 6F_{k-2}}{k(F_k + 2F_{k-1}) + 4F_k - 5F_{k-1}}. \tag{3.17}$$

3.4. Asymptotic behaviour

According to (3.9) and (2.26), the asymptotic number of animals with area k on a double staircase is such that

$$\theta = -1 \qquad \lambda = \phi. \tag{3.18}$$

From (3.13) we deduce that

$$\bar{l}(k) = \frac{k}{2+\phi} + \mathrm{O}(1) = \frac{5-\sqrt{5}}{10}k + \mathrm{O}(1) \tag{3.19}$$

whereas (3.17) leads to

$$\overline{h}(k) = 5\frac{1+\phi}{2+\phi} + \mathrm{O}(k^{-1}) = \frac{5+\sqrt{5}}{2} + \mathrm{O}(k^{-1}). \tag{3.20}$$

4. Discussion

The behaviour of the number of animals $F_{k,l}$ as a function of l for $k \gg 1$ can be obtained by expandind $\ln F_{k,l}$ to second order near its maximum using the Stirling approximation. This leads to a Gaussian distribution:

$$F_{k,l} \simeq \frac{5^{1/4}\phi^k}{\sqrt{2\pi k}} \exp\left[\frac{(l-\bar{l}(k))^2}{2\overline{\Delta l^2}(k)}\right]. \tag{4.1}$$

Here $\bar{l}(k)$ and $\overline{\Delta l^2(k)}$ are the leading contributions to the mean length and the mean-square deviation as given in (2.28) and (2.31), respectively. The prefactor follows from the normalization to $F_k \simeq \phi^k/\sqrt{5}$.

The same behaviour is obtained in the double staircase for which, according to (3.7), $G_{k,l} \simeq lF_{k,l} \simeq \bar{l}(k)F_{k,l}$. Only the normalization differs since, according to (3.9), $G_k \simeq k(F_k + 2F_{k-1})/5 \simeq k\phi^k/5$. Hence we have

$$G_{k,l} \simeq \sqrt{\frac{k}{2\pi\sqrt{5}}}\phi^k \exp\left[\frac{(l-\bar{l}(k))^2}{2\overline{\Delta l^2}(k)}\right]. \tag{4.2}$$

Thus, to leading order, $\bar{l}(k)$ and $\overline{\Delta l^2}(k)$ are the same for both problems. Furthermore, the Gaussian distribution leads to $\overline{h}(k) = k\overline{1/l}(k) \simeq k/\bar{l}(k)$ and we obtain the relation $\overline{h}(k)\bar{l}(k) = k$, valid to leading order too. These conclusions are in agreement with the exact results obtained in sections 2.4 and 3.4.

The animals are strongly anisotropic. Their length, measured along the staircase, scales as $\bar{l}(k) \sim k^{\nu_\parallel}$ and their transverse size as $\overline{h}(k) \sim k^{\nu}$ with $\nu_\parallel = \nu z = 1$ and $\nu = 0$. Thus the anisotropy exponent z is infinite.

In this paper we have studied column-convex animals in staircases with steps of unit height. The problem can be generalized by considering staircases with steps of constant arbitrary height. Then Fibonacci numbers are replaced by generalized Fibonacci numbers.

Appendix. Calculation of the coefficients of $F^3(z)$

Starting from the partial fraction expansion for $F(z)$ in (2.25) we have

$$F^3(z) = \frac{1}{5\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right)^3$$
$$= \frac{1}{5\sqrt{5}}\left[\frac{1}{(1-\phi z)^3} - \frac{1}{(1-\hat{\phi} z)^3} - \frac{3}{(1-\phi z)(1-\hat{\phi} z)}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right)\right]. \quad \text{(A.1)}$$

Making use of the series expansion

$$\frac{1}{(1-x)^3} = \sum_k \binom{k+2}{k} x^k = \tfrac{1}{2}\sum_{k=0}^{\infty}(k+1)(k+2)x^k \quad \text{(A.2)}$$

with $x = \phi z$ and $x = \hat{\phi} z$ in the two first terms and, in the third, taking into account the relation

$$(1-\phi z)(1-\hat{\phi} z) = 1 - z - z^2 \quad \text{(A.3)}$$

which follows from the expressions of ϕ and $\hat{\phi}$ in (2.25), we have

$$F^3(z) = \frac{1}{5\sqrt{5}}\left[\frac{1}{2}\sum_{k=0}^{\infty}(k+1)(k+2)(\phi^k - \hat{\phi}^k)z^k - 3\sqrt{5}\frac{F^2(z)}{z}\right]. \quad \text{(A.4)}$$

Furthermore, since

$$\phi^k - \hat{\phi}^k = [z^k]\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right) = \sqrt{5}F_k \quad \text{(A.5)}$$

we can rewrite (A.4) as

$$F^3(z) = \sum_{k=1}^{\infty}\frac{(k+1)(k+2)}{10}F_k z^k - \frac{3}{5}\frac{F^2(z)}{z}. \quad \text{(A.6)}$$

Thus, the coefficients of the series are given by

$$F_k^{(3)} = \frac{(k+1)(k+2)}{10}F_k - \frac{3}{5}[z^k]\frac{F^2(z)}{z}$$
$$= \frac{(k+1)(k+2)}{10}F_k - \frac{3}{5}F_{k+1}^{(2)}. \quad \text{(A.7)}$$

Using (2.11), it is easy to verify that $F_1^{(3)} = F_2^{(3)} = 0$ and $F_3^{(3)} = 1$ as expected since the series expansion of $F(z)$ starts with z.

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